

Gauss-Bonnet theorem in sub-Riemannian Heisenberg space \mathbb{H}^1

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Abstract

We prove a version of Gauss-Bonnet theorem in sub-Riemannian Heisenberg space \mathbb{H}^1 . The sub-Riemannian distance makes \mathbb{H}^1 in a metric space and consequently with a spherical Hausdorff measure. Using this measure, we define a Gaussian curvature at points of a surface S where the sub-Riemannian distribution is transverse to the tangent space of S . If all points of S have this property, we prove a Gauss-Bonnet formula and for compact surfaces (which are topologically a torus) we obtain $\int_S K = 0$.

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1 Introduction

In this paper, we prove a Gauss-Bonnet type theorem for surfaces inside Heisenberg group \mathbb{H}^1 . In this space consider a distribution D generated by vector fields

$$\mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{1}{2}y \frac{\partial}{\partial z} \quad ; \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + \frac{1}{2}x \frac{\partial}{\partial z},$$

and a scalar product in D such that $\mathbf{e}_1, \mathbf{e}_2$ are orthonormal. Complete these vector fields to a basis of left invariant vector fields in \mathbb{H}^1 introducing

$$\mathbf{e}_0 = [\mathbf{e}_1, \mathbf{e}_2] = \frac{\partial}{\partial z}.$$

Therefore, if $\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2$ are dual forms to $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$, then the volume element invariant by the group action is $dV = \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2$.

With the scalar product in D , consider the distance between two points as the infimum of length of curves tangent to D that connect them. With this distance, \mathbb{H}^1 is a metric space with Hausdorff dimension four and the differentiable surfaces have dimension three. At points of a surface S where the distribution D does not coincide with TS , the intersection $D \cap TS$ has dimension one, and we obtain a direction called *characteristic* at this point of S . We suppose every point of surface S has a characteristic direction. The vector field *normal horizontal* η is an unitary vector field in D and orthogonal to the characteristic direction which we suppose is globally defined. Given a compact set $K \subset S$, the 3-dimensional (spherical) Hausdorff measure of K is given by $\int_K i(\eta)dV$. A curve transverse to D has Hausdorff dimension two and its (spherical) Hausdorff measure is given by $\int_\gamma \mathbf{e}^0$. For more details, see [3, 4, 5, 6].

To prove a Gauss-Bonnet theorem, we need a concept of curvature of surfaces. The image by the left transport of the normal horizontal in a neighborhood of a point in S is contained in $S^1 \subset T_0\mathbb{H}^1$, therefore the normal horizontal does not suit as a Gauss map. But we can consider the 1-form η^* defined on S by $\eta^*(\eta) = 1$ and $\eta^*|_{TS} = 0$. The analogous of Gauss application is

$$\begin{aligned} g := \exp \circ L^* \circ \eta^* : S &\rightarrow \mathbb{H}^1 \\ p &\mapsto \exp(L_p^*(\eta^*(p))), \end{aligned}$$

with image in the cilinder $S^1 \times \mathbb{R}$. Then we define

$$K(p) = \lim_{U \rightarrow \{p\}} \frac{\int_{g(U)} i(\tilde{\eta}) dV}{\int_U i(\eta) dV} \quad (1)$$

as the *Gaussian curvature* of surface S at point p , where $\tilde{\eta}$ is the horizontal normal to $g(S)$.

Consider the *adapted covariant derivative* $\bar{\nabla}$, defined in [2], for which the left invariant vector fields are parallel. We then define a covariant derivative ∇ on S by projecting $\bar{\nabla}$ in the direction of η^* ,

$$\nabla_X Y = \bar{\nabla}_X Y - \eta^*(\bar{\nabla}_X Y)\eta,$$

where X, Y are vector fields on S . The relevant fact is that the curvature associated to ∇ coincides with the one defined by Gauss map (1).

To get the local form of Gauss-Bonnet theorem, we still need the concept of geodesic curvature for curves in the surface. We consider curves transverse to characteristic directions, and for these curves we define the tangent field

$$T = \frac{\gamma'}{\mathbf{e}^0(\gamma')}.$$

If N is an unitary field in the characteristic direction along the transverse curve γ , with orientation conveniently chosen, then we have $\nabla_T T = kN$, and k is the *curvature* of γ . Finally, to characterize the variation of directions of two transverse curves by a same vertex, we define the *corner area* between two tangent vectors of S in a point by

$$\text{ca}(v, w) = \frac{dV(\eta, v, w)}{\mathbf{e}^0(v)\mathbf{e}^0(w)}.$$

With these preliminaries, we state

Theorem 1.1 (*Gauss-Bonnet formula*) *Let R be a region contained in a coordinate domain U of S such that $T_p S \neq D_p$, for all $p \in U$, let the bounding curve γ of R be a simple closed transverse curve, and let ca_1, \dots, ca_r be the exterior corner areas of γ . Then*

$$\int_{\gamma} k + \sum_{j=1}^r ca_j + \int_R K = 0,$$

where k is the curvature function on γ and K is the Gaussian curvature function on R .

If the surface S is compact and oriented, then there exists a characteristic no-null vector field on S , therefore S is diffeomorphic to a torus. In this case, we obtain the corollary:

Corollary 1.1 *Suppose S is a differentiable compact surface in \mathbb{H}^1 such that $T_p S \neq D_p$, for all $p \in S$. Then*

$$\int_S K = 0.$$

2 The Heisenberg group

We denote by \mathbb{H}^1 the Heisenberg nilpotent Lie group whose manifold is \mathbb{R}^3 , with Lie algebra $H^1 = V_1 \oplus V_2$, $\dim V_1 = 2$, $\dim V_2 = 1$, and

$$[V_1, V_1] = V_2 \quad ; \quad [V_1, V_2] = [V_2, V_2] = 0.$$

Since \mathbb{H}^1 is nilpotent, the exponential map $\exp : H^1 \rightarrow \mathbb{H}^1$ is a diffeomorphism. Let be $\mathbf{e}_1, \mathbf{e}_2$ a basis of V_1 and $\mathbf{e}_0 = [\mathbf{e}_1, \mathbf{e}_2] \in V_2$. By applying the Baker-Campbell-Hausdorff formula we have

$$\exp^{-1}(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y].$$

Since $[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_0$, writing $X = x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_0$, and $Y = x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_0$, we get

$$X + Y + \frac{1}{2}[X, Y] = (x_1 + x_2)\mathbf{e}_1 + (y_1 + y_2)\mathbf{e}_2 + (z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1))\mathbf{e}_0.$$

We identify \mathbb{H}^1 with \mathbb{R}^3 by identifying (x, y, z) with $\exp(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_0)$, and this is known as canonical coordinates of first kind or exponential coordinates. In these coordinates, the group operation is

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1)),$$

the exponential is

$$\exp(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_0) = (x, y, z),$$

and the left invariant vector fields $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0$ are given by

$$\begin{cases} \mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \\ \mathbf{e}_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \\ \mathbf{e}_0 = \frac{\partial}{\partial z}, \end{cases}$$

with brackets $[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_0$, and $[\mathbf{e}_0, \mathbf{e}_1] = [\mathbf{e}_0, \mathbf{e}_2] = 0$. The dual basis is

$$\begin{cases} \mathbf{e}^1 = dx, \\ \mathbf{e}^2 = dy, \\ \mathbf{e}^0 = dz + \frac{1}{2}(ydx - xdy), \end{cases}$$

with $d\mathbf{e}^0 = -\mathbf{e}^1 \wedge \mathbf{e}^2$, $d\mathbf{e}^1 = d\mathbf{e}^2 = 0$. For more details, see [1].

We identify naturally $T\mathbb{H}^1$ with $T^*\mathbb{H}^1$ by $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_0$ with $a\mathbf{e}^1 + b\mathbf{e}^2 + c\mathbf{e}^0$, and through this identification we identify H^1 with $(H^1)^*$. Therefore we can define the exponential map on the dual by

$$\begin{aligned} \exp : (H^1)^* &\rightarrow \mathbb{H}^1 \\ x\mathbf{e}^1 + y\mathbf{e}^2 + z\mathbf{e}^0 &\mapsto (x, y, z). \end{aligned}$$

The left translation is defined by

$$L_{(x,y,z)}(x_1, y_1, z_1) = (x, y, z)(x_1, y_1, z_1),$$

and

$$L_{(x,y,z)}^{-1} = L_{(-x,-y,-z)}.$$

Let be $D \subset T\mathbb{H}^1$ the two-dimensional distribution generated by the vector fields $\mathbf{e}_1, \mathbf{e}_2$, so that D is the null space of \mathbf{e}^0 . On D , we define a scalar product $\langle \cdot, \cdot \rangle$, such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis of D . An operator $J : D \rightarrow D$ is well-defined by

$$J(a\mathbf{e}_1 + b\mathbf{e}_2) = -b\mathbf{e}_1 + a\mathbf{e}_2.$$

The *element of volume* dV in \mathbb{H}^1 is $dV = \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 = dx \wedge dy \wedge dz$. A differentiable curve $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{H}^1$ is *transversal* if $\mathbf{e}^0(\gamma'(t)) \neq 0$, for every $t \in [a, b]$. We say that a transversal curve γ is *unitarily parametrized* if $|\mathbf{e}^0(\gamma'(t))| = 1$, for every $t \in [a, b]$.

3 The adapted covariant derivative

If X, Y are vector fields on \mathbb{H}^1 , we define the adapted covariant derivative introduced in [2] by:

$$\bar{\nabla}_X Y = \sum_{j=0}^2 db_j(X) \mathbf{e}_j,$$

where $Y = b_0 \mathbf{e}_0 + b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2$. Then $\bar{\nabla}$ is null at left invariant vector fields on \mathbb{H}^1 .

Proposition 3.1 *The covariant derivative $\bar{\nabla}$ has the following properties:*

1. If $Y \in \underline{D}$, then $\bar{\nabla}_X Y \in \underline{D}$ for all $X \in \underline{T\mathbb{H}^1}$;
2. If $Y, Z \in \underline{D}$, then

$$\bar{\nabla}_X \langle Y, Z \rangle = \langle \bar{\nabla}_X Y, Z \rangle + \langle Y, \bar{\nabla}_X Z \rangle,$$

for all $X \in \underline{T\mathbb{H}^1}$;

3. The torsion \bar{T} of $\bar{\nabla}$ is

$$\bar{T} = -\mathbf{e}^1 \wedge \mathbf{e}^2 \otimes \mathbf{e}_0 = d\mathbf{e}^0 \otimes \mathbf{e}_0,$$

4. The curvature \bar{K} of $\bar{\nabla}$ is null.

Proof. We shall proceed the proof of 3, the others being similar. If $X = \sum_{i=0}^2 a_i \mathbf{e}_i$ and $Y = \sum_{j=0}^2 b_j \mathbf{e}_j$, then

$$\bar{T}(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = -(a_1 b_2 - a_2 b_1) \mathbf{e}_0 = -\mathbf{e}^1 \wedge \mathbf{e}^2(X, Y) \mathbf{e}_0 = d\mathbf{e}^0(X, Y) \mathbf{e}_0.$$

□

Observe that the covariant derivative in the cotangent bundle $(T\mathbb{H}^1)^*$ satisfies $\bar{\nabla} \mathbf{e}^i = 0$, for $i = 0, 1, 2$.

4 Surfaces in \mathbb{H}^1

Suppose S is an oriented differentiable two-dimensional manifold in \mathbb{H}^1 . Note that $\dim(D \cap TS) \geq 1$, and, since $d\mathbf{e}^0 = \mathbf{e}^1 \wedge \mathbf{e}^2$, the set of points where the tangent space of S coincides with the distribution has empty interior. We denote by Σ this set and by S' its complement on S ,

$$\Sigma = \{x \in S : \dim(D_x \cap T_x S) = 2\} \quad ; \quad S' = S - \Sigma.$$

The set S' is open in S . In what follows we will suppose $\Sigma = \emptyset$, so $S = S'$. With this hypothesis on S , the one-dimensional vector subbundle $D \cap TS$ is well defined. Suppose $U \subset S$ is an open set such that we can define a unitary vector field f_1 with values in $D \cap TS$, so $\langle f_1, f_1 \rangle = 1$.

Definition 4.1 *The unitary vector field $\eta \in \underline{D}$ defined by*

$$\eta = -Jf_1$$

is the horizontal normal to S .

Then we can define $\eta^* \in (T\mathbb{H}^1)^*|_S$ by

$$\eta^*(\eta) = 1 \quad ; \quad \eta(TS) = 0.$$

We call η^* the *horizontal conormal* to S .

Definition 4.2 *The application*

$$\begin{aligned} g := \exp \circ L^* \circ \eta^* : S &\rightarrow \mathbb{H}^1 \\ p &\mapsto \exp(L_p^*(\eta^*(p))) \end{aligned}$$

is the Gauss map of S .

Let be

$$f_2 = \mathbf{e}_0 - \eta^*(\mathbf{e}_0)\eta.$$

Then $\{f_1, f_2\}$ is a *special* basis of TS on the open set U . If

$$\eta = \cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2,$$

for some real function α on U , reducing U if necessary, then

$$f_1 = -\sin \alpha \mathbf{e}_1 + \cos \alpha \mathbf{e}_2,$$

and, if we denote by $A = -\eta^*(\mathbf{e}_0)$, we write

$$f_2 = \mathbf{e}_0 + A\eta.$$

The dual basis of $(T\mathbb{H}^1)^*$ on S is

$$\begin{cases} \eta^* = \cos \alpha \mathbf{e}^1 + \sin \alpha \mathbf{e}^2 - A\mathbf{e}^0, \\ f^1 = -\sin \alpha \mathbf{e}^1 + \cos \alpha \mathbf{e}^2, \\ f^2 = \mathbf{e}^0. \end{cases}$$

The inverse relations are

$$\begin{cases} \mathbf{e}^0 = f^2, \\ \mathbf{e}^1 = \cos \alpha \eta^* - \sin \alpha f^1 + A \cos \alpha f^2, \\ \mathbf{e}^2 = \sin \alpha \eta^* + \cos \alpha f^1 + A \sin \alpha f^2, \end{cases}$$

and

$$\mathbf{e}^1 \wedge \mathbf{e}^2 = \eta^* \wedge f^1 - A f^1 \wedge f^2.$$

Also, it follows

$$\begin{cases} df^1 = -d\alpha \wedge \eta^* - A d\alpha \wedge f^2, \\ df^2 = -\eta^* \wedge f^1 + A f^1 \wedge f^2, \\ d\eta^* = (d\alpha + A^2 f^2 + A\eta^*) \wedge f^1 - dA \wedge f^2, \end{cases}$$

and, since $\eta^* = 0$ on S , we get

$$\begin{cases} df^1 = -A d\alpha \wedge f^2, \\ df^2 = A f^1 \wedge f^2, \\ 0 = (d\alpha + A^2 f^2) \wedge f^1 - dA \wedge f^2. \end{cases}$$

From this last relation, we obtain

$$d\alpha(f_2) = -(dA(f_1) + A^2).$$

Definition 4.3 *The element of area in S is*

$$i(\eta)dV.$$

Since $dV = \eta^* \wedge f^1 \wedge f^2$, then $dS = f^1 \wedge f^2$. Let's find the area of $g(R)$ for a region $R \subset S$. Observe that, for all $p \in S$,

$$g(p) = (\cos \alpha(p), \sin \alpha(p), -A(p)).$$

Then $g(R)$ is contained on the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$. The tangent space TC is generated by

$$\begin{cases} -y\mathbf{e}_1 + x\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_0 & (= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) \\ \mathbf{e}_0 & (= \frac{\partial}{\partial z}). \end{cases}$$

It follows that

$$\begin{cases} \tilde{f}_1 = -y\mathbf{e}_1 + x\mathbf{e}_2 \\ \tilde{f}_2 = \mathbf{e}_0, \end{cases}$$

so

$$\tilde{\eta} = -J(\tilde{f}_1) = x\mathbf{e}_1 + y\mathbf{e}_2.$$

The element of area on C is $d\tilde{S} = \tilde{f}^1 \wedge \tilde{f}^2$. Then

$$\begin{aligned} \text{Area}(g(R)) &= \int_{g(R)} d\tilde{S} = \int_{g(R)} \tilde{f}^1 \wedge \tilde{f}^2 = \int_R g^*(\tilde{f}^1 \wedge \tilde{f}^2) = \int_R g^*(\tilde{f}^1 \wedge \tilde{f}^2)(f_1, f_2) f^1 \wedge f^2 \\ &= \int_R (\tilde{f}^1 \wedge \tilde{f}^2)(g_* f_1, g_* f_2) dS. \end{aligned}$$

Now,

$$\begin{aligned}
dg &= -\sin \alpha d\alpha \otimes \frac{\partial}{\partial x} + \cos \alpha d\alpha \otimes \frac{\partial}{\partial y} - dA \otimes \frac{\partial}{\partial z} \\
&= -\sin \alpha d\alpha \otimes (\mathbf{e}_1 + \frac{1}{2} \sin \alpha \mathbf{e}_0) + \cos \alpha d\alpha \otimes (\mathbf{e}_2 - \frac{1}{2} \cos \alpha \mathbf{e}_0) - dA \otimes \mathbf{e}_0 \\
&= d\alpha \otimes \tilde{f}_1 - (\frac{1}{2} d\alpha + dA) \otimes \tilde{f}_2,
\end{aligned}$$

and so

$$\begin{aligned}
(\tilde{f}^1 \wedge \tilde{f}^2)(g_* f_1, g_* f_2) &= -d\alpha(f_1) \left(\frac{1}{2} d\alpha(f_2) + dA(f_2) \right) + d\alpha(f_2) \left(\frac{1}{2} d\alpha(f_1) + dA(f_1) \right) \\
&= -d\alpha \wedge dA(f_1, f_2).
\end{aligned}$$

We just proved that

$$\text{Area}(g(R)) = \int_R -d\alpha \wedge dA(f_1, f_2) dS.$$

As $\text{Area}(R) = \int_R dS$, we obtain from (1) that

$$K = -d\alpha \wedge dA(f_1, f_2).$$

Proposition 4.1 *The Gaussian curvature K of S is given by*

$$K = -d\alpha \wedge dA(f_1, f_2).$$

5 The projection of $\overline{\nabla}$ by η^*

Given $X, Y \in \underline{TS}$, we define

$$\nabla_X Y = \overline{\nabla}_X Y - \eta^*(\overline{\nabla}_X Y)\eta.$$

Proposition 5.1 *The operator ∇ is a covariant derivative in TS , and satisfies:*

1. $\nabla f_1 = 0$;
2. $\nabla f_2 = A d\alpha \otimes f_1$;
3. $\nabla f^1 = -A d\alpha \otimes f^2$;
4. $\nabla f^2 = 0$.

Proof. It is clear that, if $X, Y \in TS$, then $\nabla_X Y \in TS$, $\nabla_X Y$ is linear on X and additive on Y . Furthermore, if f is a real function on S , we have

$$\nabla_X fY = df(X)Y + f\overline{\nabla}_X Y - \eta^*(df(X)Y + f\overline{\nabla}_X Y)\eta = df(X)Y + f\nabla_X Y.$$

Finally,

1. $\overline{\nabla}_X f_1 = \overline{\nabla}_X(-\sin \alpha \mathbf{e}_1 + \cos \alpha \mathbf{e}_2) = d\alpha(X)(-\cos \alpha \mathbf{e}_1 - \sin \alpha \mathbf{e}_2) = -d\alpha(X)\eta$, so $\nabla_X f_1 = 0$.
2. $\overline{\nabla}_X f_2 = \overline{\nabla}_X(\mathbf{e}_0 + A\eta) = dA(X)\eta + A\overline{\nabla}_X(\cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2) = dA(X)\eta + A d\alpha(X)(-\sin \alpha \mathbf{e}_1 + \cos \alpha \mathbf{e}_2) = dA(X)\eta + A d\alpha(X)f_1$, so $\nabla_X f_2 = A d\alpha(X)f_1$.

3. $(\nabla_X f^1)(f_1) = -f^1(\nabla_X f_1) = 0$ and $(\nabla_X f^1)(f_2) = -f^1(\nabla_X f_2) = -A d\alpha(X)$ so $\nabla_X f^1 = -A d\alpha(X) f^2$.
4. $(\nabla_X f^2)(f_1) = -f^2(\nabla_X f_1) = 0$ and $(\nabla_X f^2)(f_2) = -f^2(\nabla_X f_2) = 0$ so $\nabla_X f^2 = 0$.

□

It follows from this proof that $\overline{\nabla}_X \eta = \nabla_X \eta = d\alpha(X) f_1$ and $\overline{\nabla}_X \eta^* = d\alpha(X) f^1 - dA(X) f^2$, for $X \in TS$.

Definition 5.1 *The covariant derivative ∇ is the adapted covariant derivative on S .*

Proposition 5.2 *The torsion T of ∇ is $T = A f^1 \wedge f^2 \otimes f_2$.*

Proof. We have

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] - \eta^*(\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y])\eta = \overline{T}(X, Y) - \eta^*(\overline{T}(X, Y))\eta,$$

so

$$T = -\mathbf{e}^1 \wedge \mathbf{e}^2 \otimes (\mathbf{e}_0 - \eta^*(\mathbf{e}_0)\eta) = A f^1 \wedge f^2 \otimes f_2.$$

Proposition 5.3 *The curvature tensor R of ∇ is $R = dA \wedge d\alpha \otimes f^2 \otimes f_1$.*

Proof. Clearly $R(X, Y) f_1 = 0$, and

$$\begin{aligned} R(X, Y) f_2 &= \nabla_X \nabla_Y f_2 - \nabla_Y \nabla_X f_2 - \nabla_{[X, Y]} f_2 \\ &= \nabla_X (Ad\alpha(Y) f_1) - \nabla_Y (Ad\alpha(X) f_1) - Ad\alpha([X, Y]) f_1 \\ &= (X(Ad\alpha(Y)) - Y(Ad\alpha(X)) - Ad\alpha([X, Y])) f_1 \\ &= d(Ad\alpha)(X, Y) f_1. \end{aligned}$$

□

Proposition 5.4 *The Gaussian curvature K is given by*

$$K = \langle R(f_1, f_2) f_2, f_1 \rangle = dA \wedge d\alpha(f_1, f_2). \quad (2)$$

6 The second fundamental form

From the equation

$$\overline{\nabla}_X Y = \nabla_X Y + \eta^*(\overline{\nabla}_X Y)\eta = \nabla_X Y - (\overline{\nabla}_X \eta^*)(Y)\eta,$$

for $X, Y \in \underline{TS}$, we define a bilinear form $V: TS \times TS \rightarrow \mathbb{R}$:

Definition 6.1 *The bilinear form $V: TS \times TS \rightarrow \mathbb{R}$, defined by*

$$V(X, Y) = -(\overline{\nabla}_X \eta^*)(Y)$$

is the second fundamental form associated to S .

From

$$\overline{\nabla}\eta^* = \overline{\nabla}(\cos \alpha \mathbf{e}^1 + \sin \alpha \mathbf{e}^2 - A\mathbf{e}^0) = d\alpha \otimes (-\sin \alpha \mathbf{e}^1 + \cos \alpha \mathbf{e}^2) - dA \otimes \mathbf{e}^0,$$

we get

$$V(X, Y) = -d\alpha(X)f^1(Y) + dA(X)f^2(Y).$$

The second fundamental form is not symmetric in general. In fact, for $X, Y \in TS$,

$$\begin{aligned} V(X, Y) - V(Y, X) &= -(\overline{\nabla}_X \eta^*)(Y) + (\overline{\nabla}_Y \eta^*)(X) = \eta^*(\overline{\nabla}_X Y) - \eta^*(\overline{\nabla}_Y X) \\ &= \eta^*(\overline{T}(X, Y)) = d\mathbf{e}^0(X, Y)\eta^*(\mathbf{e}_0) = -Adf^2(X, Y) \\ &= -A^2 f^1 \wedge f^2(X, Y). \end{aligned}$$

Theorem 6.1 *The curvature K and the second fundamental form V satisfy:*

1. (Gauss equation) $K(X, Y)Z = (-d\alpha(X)V(Y, Z) + d\alpha(Y)V(X, Z))f_1$;
2. (Codazzi equation) $\nabla_X V(Y, Z) - \nabla_Y V(X, Z) + V(T(X, Y), Z) = 0$.

Proof. By applying the definition of curvature, we obtain

$$\begin{aligned} K(X, Y)Z &= \overline{\nabla}_X(\overline{\nabla}_Y Z - V(Y, Z)\eta) - V(X, \nabla_Y Z)\eta - \overline{\nabla}_Y(\overline{\nabla}_X Z - V(X, Z)\eta) \\ &\quad + V(Y, \nabla_X Z)\eta - \overline{\nabla}_{[X, Y]}Z + V([X, Y], Z)\eta \\ &= \overline{K}(X, Y)Z - X(V(Y, Z))\eta - V(Y, Z)d\alpha(X)f_1 + Y(V(X, Z))\eta \\ &\quad + V(X, Z)d\alpha(Y)f_1 - V(X, \nabla_Y Z)\eta + V(Y, \nabla_X Z)\eta + V([X, Y], Z)\eta \\ &= (-\nabla_X V(Y, Z) - V(\nabla_X Y, Z) + \nabla_Y V(X, Z) + V(\nabla_Y X, Z) + V([X, Y], Z))\eta \\ &\quad + (-V(Y, Z)d\alpha(X) + V(X, Z)d\alpha(Y))f_1 \\ &= -(\nabla_X V(Y, Z) - \nabla_Y V(X, Z) + V(\nabla_X Y - \nabla_Y X - [X, Y], Z))\eta \\ &\quad - (V(Y, Z)d\alpha(X) - V(X, Z)d\alpha(Y))f_1. \end{aligned}$$

Since $K(X, Y)Z \in TS$, we obtain

$$K(X, Y)Z = -(V(Y, Z)d\alpha(X) - V(X, Z)d\alpha(Y))f_1,$$

and

$$\nabla_X V(Y, Z) - \nabla_Y V(X, Z) + V(T(X, Y), Z) = 0.$$

□

7 Curvature of transverse curves in the surface S

Let be $\gamma: [a, b] \subset \mathbb{R} \rightarrow S$ a differentiable curve such that $\gamma'(t)$ is transversal, i.e., $f^2(\gamma'(t)) \neq 0$ for all $t \in [a, b]$. Let be T defined by

$$T(t) = \frac{1}{|f^2(\gamma'(t))|} \gamma'(t),$$

the unitary tangent field along γ . As $f^2(T(t)) = \pm 1$, then

$$\nabla_T f^2(T) + f^2(\nabla_T T) = 0,$$

and as $\nabla f^2 = 0$, we know that $\nabla_T T$ is a multiple of f_1 . We write

$$\nabla_T T = kN,$$

where the vector field $N = \epsilon f_1$ on γ , and $\epsilon = +1$ if $\{T, f_1\}$ is positively oriented and $\epsilon = -1$, otherwise. Observe that $\epsilon f^2(T) < 0$. The function $k: [a, b] \rightarrow \mathbb{R}$ is the *curvature* of γ .

Definition 7.1 *The function $k = \langle \nabla_T T, N \rangle$ is the curvature of the transverse curve γ .*

Proposition 7.1 *The curvature k is given by*

$$k = \frac{\epsilon}{f^2(\gamma')} \left(\frac{d}{dt} \frac{f^1(\gamma')}{f^2(\gamma')} + A d\alpha(\gamma') \right).$$

Proof. It follows from the definition that $k = \epsilon f^1(\nabla_T T)$, so

$$k = \frac{\epsilon}{|f^2(\gamma')|} f^1(\nabla_{\gamma'}(\frac{1}{|f^2(\gamma')|} \gamma')) = \frac{\epsilon}{|f^2(\gamma')|} \left(\nabla_{\gamma'}(f^1(\frac{1}{|f^2(\gamma')|} \gamma')) - (\nabla_{\gamma'} f^1)(\frac{1}{|f^2(\gamma')|} \gamma') \right),$$

and the proposition follows. □

8 Gauss-Bonnet theorem

In this section, let be $R \subset S$ a fundamental set, and c a fundamental 2-chain such that $|c| = R$. The oriented curve $\gamma = \partial c$ is the bounding curve of R . The curve γ is piecewise differentiable, and composed of differentiable curves $\gamma_j: [s_j, s_{j+1}] \rightarrow S$, $j = 1, \dots, r$, with $\gamma_j(s_{j+1}) = \gamma_{j+1}(s_{j+1})$, for $j = 1, \dots, r-1$ and $\gamma_1(s_1) = \gamma_r(s_{r+1})$. We define the corner area at the *vertices* $\gamma_j(s_{j+1})$ as $ca_j = \text{ca}(\gamma'_j(s_{j+1}), \gamma'_{j+1}(s_{j+1}))$, $j = 1, \dots, r-1$ and $ca_r = \text{ca}(\gamma'_r(s_{r+1}), \gamma'_1(s_1))$.

Theorem 8.1 (*Gauss-Bonnet formula*) *Let R be contained in a coordinate domain U of S , let the bounding curve γ of R be a simple closed transverse curve, and let ca_1, \dots, ca_r be the exterior corner areas of γ . Then*

$$\int_{\gamma} k + \sum_{j=1}^r ca_j + \int_R K = 0,$$

where k is the curvature function on γ and K is the scalar curvature function on R .

Proof. Let $\gamma_1, \dots, \gamma_r$ be the C^∞ pieces of γ with γ_j defined on the interval $[s_j, s_{j+1}]$, with $\gamma_j(s_{j+1}) = \gamma_{j+1}(s_{j+1})$, for $j = 1, \dots, r-1$, and $\gamma_r(s_{r+1}) = \gamma_1(s_1)$. Let be $ca_j = \text{ca}(\gamma'_j(s_{j+1}), \gamma'_{j+1}(s_{j+1}))$, for $j = 1, \dots, r-1$ and $ca_r = \text{ca}(\gamma'_r(s_{r+1}), \gamma'_1(s_1))$. In each C^∞ piece of γ we have the positive orientation T and the curvature $\nabla_T T = \epsilon k f_1$. Then from (2), Propositions 5.1 and 7.1, and

$$f_2 = \frac{1}{f^2(\gamma'_j)} \gamma'_j - \frac{f^1(\gamma'_j)}{f^2(\gamma'_j)} f_1 = -\epsilon T - \frac{f^1(\gamma'_j)}{f^2(\gamma'_j)} f_1,$$

since $\epsilon = -\frac{|f^2(\gamma'_j)|}{f^2(\gamma'_j)}$, we obtain

$$\begin{aligned}
\int_R K &= \int_c K f^1 \wedge f^2 = \int_c dA \wedge d\alpha(f_1, f_2) f^1 \wedge f^2 = \int_c dA \wedge d\alpha = \int_{\partial c} A d\alpha \\
&= \sum_{j=1}^r \int_{[s_j, s_{j+1}]} A d\alpha(\gamma'_j) \\
&= \sum_{j=1}^r \int_{[s_j, s_{j+1}]} \left(\epsilon f^2(\gamma'_j) k - \frac{d f^1(\gamma'_j)}{dt f^2(\gamma'_j)} \right) \\
&= \int_{\partial c} -k |f^2| - \sum_{j=1}^r \left(\frac{f^1(\gamma'_j(s_{j+1}))}{f^2(\gamma'_j(s_{j+1}))} - \frac{f^1(\gamma'_j(s_j))}{f^2(\gamma'_j(s_j))} \right) \\
&= - \int_{\gamma} k - \sum_{j=1}^r \frac{f^1(\gamma'_j(s_{j+1}))}{f^2(\gamma'_j(s_{j+1}))} + \sum_{j=0}^{r-1} \frac{f^1(\gamma'_{j+1}(s_{j+1}))}{f^2(\gamma'_{j+1}(s_{j+1}))} \\
&= - \int_{\gamma} k + \sum_{j=1}^{r-1} \left(\frac{f^1(\gamma'_{j+1}(s_{j+1}))}{f^2(\gamma'_{j+1}(s_{j+1}))} - \frac{f^1(\gamma'_j(s_{j+1}))}{f^2(\gamma'_j(s_{j+1}))} \right) + \left(\frac{f^1(\gamma'_1(s_1))}{f^2(\gamma'_1(s_1))} - \frac{f^1(\gamma'_r(s_{r+1}))}{f^2(\gamma'_r(s_{r+1}))} \right) \\
&= - \int_{\gamma} k + \sum_{j=1}^{r-1} \text{ca}(\gamma'_{j+1}(s_{j+1}), \gamma'_j(s_{j+1})) + \text{ca}(\gamma'_1(s_1), \gamma'_r(s_{r+1})) \\
&= - \int_{\gamma} k - \sum_{j=1}^r \text{ca}_j.
\end{aligned}$$

If the surface S is compact and oriented, then there exists a characteristic no-null vector field on S , therefore S is diffeomorphic to a torus. In this case, we obtain the corollary:

Corollary 8.1 *Suppose S is a differentiable compact surface in \mathbb{H}^1 with $\Sigma = \emptyset$. Then*

$$\int_S K = 0.$$

Proof. In fact, we can triangulate S by a finite number of triangles Δ_i , $i = 1, \dots, s$, such that the boundary of each Δ_i is composed by transverse curves. As the triangles are positively oriented, then

$$\int_S K = \sum_{i=1}^s \int_{\Delta_i} K = - \sum_{i=1}^s \int_{\partial \Delta_i} k - \sum_{i=1}^s \sum_{r=1}^3 \text{ca}_{ir},$$

where $\partial \Delta_i$ is the boundary of Δ_i positively oriented, and ca_{ir} , $r \in \{1, 2, 3\}$, are the corner areas at each vertex of Δ_i . If Δ_i and Δ_l have sides Δ_{iu} and Δ_{lv} in common, they have opposite orientations, so $\int_{\Delta_{iu}} k + \int_{\Delta_{lv}} k = 0$; therefore, $\sum_{i=1}^s \int_{\partial \Delta_i} k = 0$. In the same way, at a common vertex, the corner areas sum null, so $\sum_{i=1}^s \sum_{r=1}^3 \text{ca}_{ir} = 0$, and the proposition is proved. □

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